The Power Spectral Density and the Autocorrelation

The autocorrelation of a real, stationary signal $x(t)$ is defined to by $R_x(\tau) = E[x(t)\,x(t+\tau)]$. The Fourier transform of $R_x(\tau)$ is called the Power Spectral Density (PSD) $S_x(f)$. Thus:

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-2\pi i f \tau}\, d\tau\,. \tag{1}$$

The question is: what *is* the PSD? What does it mean? What is a "spectral density," and why is S_x called a *power* spectral density?

To answer this question, recall that

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi i f t}\, dt\,. \tag{2}$$

To avoid convergence problems, we consider only a version of the signal observed over a finite-time T,[1] $x_T = x(t)w_T(t)$, where

$$w_T = \begin{cases} 1 & \text{for } 0 \le |t| \le \frac{T}{2} \\ 0 & \text{for } |t| > \frac{T}{2}\,. \end{cases} \tag{3}$$

Then x_T has the Fourier transform

$$\begin{aligned} X_T(f) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\, e^{-2\pi i f t}\, dt \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-2\pi i f t}\, dt \end{aligned} \tag{4}$$

[1] This restriction is necessary because not all of our signals will be *square integrable*. However, they will be *mean square integrable*, which is what we will take advantage of here.

and so

$$
\begin{aligned}
X_T X_T^* &= \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-2\pi i f t}\, dt \right] \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(s)\, e^{2\pi i f s}\, ds \right] \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x(s)\, e^{-2\pi i f (t-s)}\, dt\, ds\,,
\end{aligned}
\tag{5}
$$

where the star denotes complex conjugation and for compactness the frequency argument of X_T has been suppressed. Taking the expectation of both sides of Eq. (5)[2]

$$
E[X_T X_T^*] = \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[x(t)\, x(s)]\, e^{-2\pi i f (t-s)}\, dt\, ds\,. \tag{6}
$$

Letting $s = t + \tau$, one sees that $E[x(t)\, x(s)] \equiv E[x(t)\, x(t+\tau)] = R_x(\tau)$, and thus

$$
E[X_T X_T^*] = \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(\tau)\, e^{-2\pi i f \tau}\, dt\, ds\,. \tag{7}
$$

To actually evaluate the above integral, the *both* variables of integration must be changed. Let

$$
\begin{aligned}
&\tau = f(t,s) = s - t \quad \text{(as already defined for Eq. 7)} \\
&\eta = g(t,s) = s + t\,.
\end{aligned}
\tag{8}
$$

Then, the integral of Eq. (7) is transformed (except for the limits of integration) using the *change of variables formula*:[3]

$$
\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(\tau)\, e^{-2\pi i f \tau}\, dt\, ds = \int \int R_x(\tau)\, e^{-2\pi i f \tau}\, |J|^{-1} d\eta\, d\tau\,, \tag{9}
$$

[2]To understand what this means, remember that Eq. (5) holds for any $x(t)$. So imagine computing Eq. (6) for different $x(t)$ obtained from different experiments on the same system (each one of these is called a *sample function)*. The expectation is over all possible sample functions. Since the exponential kernel inside the integral of Eq. (6) is the same for each sample function, it can be pulled outside of the expectation.

[3]This is a basic result from multivariable calculus. See, for example, I.S. Sokolnikoff and R.M. Redheffer, *Mathematics of Physics and Modern Engineering*, 2nd edition, McGraw-Hill, New York, 1966.

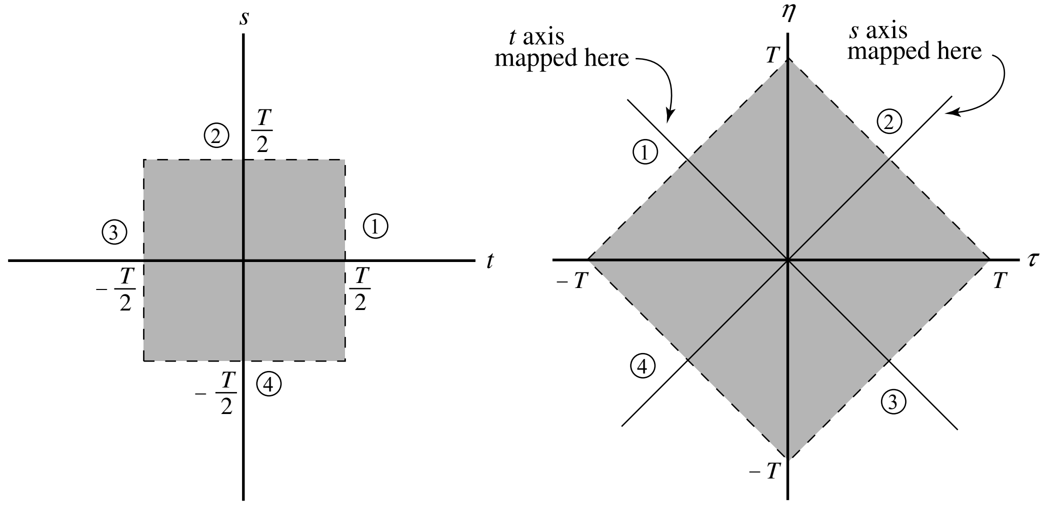

Figure 1: The domain of integration (gray regions) for the Fourier transform of the autocorrelation Eq. (7): (left) for the original variables, t and s; (right) for the transformed variables, η and τ, obtained by the change of variables Eq. (8). Notice that the square region on the left is not only rotated (and flipped about the t axis), but its area is increased by a factor of $|J| = 2$. The circled numbers show where the sides of the square on the left are mapped by the change of variables. The lines into which the t and s axes are mapped are also shown.

where $|J|$ is the absolute value of the Jacobian for the change of variables Eq. (8) given by

$$J = \begin{vmatrix} \frac{\partial f}{\partial t} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial t} & \frac{\partial g}{\partial s} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2\,. \tag{10}$$

To determine the limits of integration needed for the right hand side of Eq. (9), we need to refer to Fig. 1, in which the domain of integration is plotted in both the original (t, s) variables and the transformed (τ, η) variables. Since we wish to integrate on η first, we hold τ fixed. For $\tau > 0$, a vertical cut through the diamond-shaped region in Fig. 1 (right) shows that $-T + \tau \leq \eta \leq T - \tau$, whereas for $\tau < 0$ one finds that $-T - \tau \leq \eta \leq T + \tau$. Putting this all together yields:

$$
\begin{aligned}
E[X_T X_T^*] &= \frac{1}{2}\int_{-T}^{T}\int_{-(T-|\tau|)}^{T-|\tau|} R_x(\tau)\; e^{-2\pi i f\tau}\; d\eta\, d\tau \\
&= T\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right] R_x(\tau)\; e^{-2\pi i f\tau}\; d\tau\,.
\end{aligned}
\tag{11}
$$

Finally, dividing both sides of Eq. (11) by T and taking the limit as $T \to \infty$ gives

$$
\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}E[X_T X_T^*] &= \lim_{T\to\infty}\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right] R_x(\tau)\; e^{-2\pi i f\tau}\; d\tau \\
&= \lim_{T\to\infty}\int_{-T}^{T} R_x(\tau)\; e^{-2\pi i f\tau}\; d\tau \\
&= \int_{-\infty}^{\infty} R_x(\tau)\; e^{-2\pi i f\tau}\; d\tau \\
&= S_x(f)\,.
\end{aligned}
\tag{12}
$$

Thus, in summary, the above demonstrates that

$$
S_x(f) = \lim_{T\to\infty}\frac{1}{T}E[\,|X_T(f)|^2\,]\,. \tag{13}
$$

Recalling that $X_T(f)$ has units SU/Hz (where SU stands for "signal units," i.e., whatever units the signal $x_T(t)$ has), it is clear that $E[\,|X_T(f)|^2\,]$ has units $(\mathrm{SU/Hz})^2$. However, $1/T$ has units of Hz, so that Eq. (13) shows that the PSD has units of $(\mathrm{SU}^2)/\mathrm{Hz}$. (Of course, the units can also be determined by examining the definition of Eq. 1.)

Although it is not always literally true, in many cases the mean square of the signal is proportional to the amount of *power* in the signal.[4] The fact that

[4] This comes primarily from the fact that, in electrical circuits, the power can be written in terms of the voltage as V^2/Z, or in terms of the current as I^2Z, where Z is the circuit impedance. Thus, for electrical signals, it precisely true that the mean square of the signal will be proportional to the power. Be forewarned, however, that the mean square of the scaled signal, expressed in terms of the actual measured variable (such as displacement or acceleration), will *not* in general be equal to the average *mechanical* power in the structure being measured.

S_x is therefore interpreted has having units of "power" per unit frequency explains the name *Power* Spectral Density.

Notice that power at a frequency f_0 that does not repeatedly reappear in $x_T(t)$ as $T \to \infty$ will result in $S_x(f_0) \to 0$, because of the division by T in Eq. (13). In fact, based on this idealized mathematical definition, any signal of finite duration (or, more generally, any mean square integrable signal), *will have power spectrum identical to zero*! In practice, however, we do not let T extend much past the support $[T_{min}, T_{max}]$ of $x_T(t)$ ($T_{min/max}$ is the minimum (respectively, maximum) T for which $x_T(t) \neq 0$). Since all signals that we measure in the laboratory have the form $y(t) = x(t) + n(t)$, where $n(t)$ is broadband noise, extending T to infinity for any signal with finite support will end up giving $S_x \approx S_n$.

We conclude by mentioning some important properties of S_x. First, since S_x is an average of the magnitude squared of the Fourier transform, $S_x(f) \in \mathbb{R}$ and $S_x(f) \geq 0$ for all f. A simple change of variables in the definition Eq. (1) shows that $S_x(-f) = S_x(f)$.

Given the definition Eq. (1), we also have the dual relationship

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{2\pi i f \tau}\, df\,. \tag{14}$$

Setting $\tau = 0$ in the above yields

$$R_x(0) = E[\,x(t)^2] = \int_{-\infty}^{\infty} S_x(f)\, df\,, \tag{15}$$

which, for a mean zero signal gives

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_x(f)\, df\,. \tag{16}$$

Finally, if we assume that $x(t)$ is *ergodic* in the autocorrelation, that is, that

$$R_x(\tau) = E[x(t)\, x(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x(t+\tau)\, dt\,, \tag{17}$$

where the last equality holds for *any* sample function $x(t)$, then Eq. (15) can be rewritten as

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)^2 \, dt = \int_{-\infty}^{\infty} S_x(f) \, df \, . \tag{18}$$

The above relationship is known as Parseval's Identity.

This last identity makes it clear that, given any two frequencies f_1 and f_2, the quantity

$$\int_{f_1}^{f_2} S_x(f) \, df \tag{19}$$

represents the portion of the average signal power contained in signal frequencies between f_1 and f_2, and hence S_x is indeed a "spectral density."