Applications of the Renewal Processes

Coursera. Stochastic Processes

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Renewal processes are an efficient tool for describing the structure that recovers after some period of time. One of the most natural applications of such processes seems to be in reliability theory, which concentrates on the ability of some system to function without failure. The extensive overview of problems and possible applications of the renewal theory in this field can be found in [1].

For instance, one might consider an engineering system, e.g., a recording studio, with parts connected by K different independent cables. Each of them is assumed to fail at random and be replaced immediately upon failure. The behaviour of k-th cable, $k = 1, \dots, K$, can then be described by the renewal process

$$S_n^{(k)} = S_{n-1}^{(k)} + \xi_n^{(k)}, \qquad n \in \mathbb{N}, \quad S_0^{(k)} = 0, \quad \xi_1^{(k)}, \xi_2^{(k)}, \dots \sim \text{i.i.d. } F^{(k)}, \quad F^{(k)}(0) = 0,$$

where the arrivals $S_n^{(k)}$ represent the failures of the cable and $\xi_n^{(k)}$ stand for the times between successive failures.

The common problem arising in this context is determining the number m of spare details necessary to assure functioning of the system up to time $t \geq 0$ with some probability $1 - \alpha$, $\alpha \in (0, 1)$. In other words, assuming that the system operates only if all cables are working, one seeks for the smallest $m \in \mathbb{N}$ such that

$$\mathbb{P}\{N_t^{(1)} + \dots + N_t^{(K)} \leq m\} \geq \alpha,$$

where $N_t^{(k)}$ denotes the number of failures of the k-th cable occurred before t, $k = 1, \dots, K$. For example, if for each $k = 1, \dots, K$ $F^{(k)}$ is the exponential distribution with rate parameter λ_k, the minimal number of spare details can be found as the smallest m satisfying

$$\sum_{j=0}^{m} \frac{\left(t \sum_{k=1}^{K} \lambda_k\right)^j \exp\left\{-t \sum_{k=1}^{K} \lambda_k\right\}}{j!} \geq \alpha.$$

Another approach to the determination of the number of replacement components is to use the analogue of the central limit theorem [2]. Namely, if for a k-th cable it is known that the average lifetime is equal to μ hours and the standard deviation is σ hours, the minimal number $m \in \mathbb{N}$ of spare details necessary to ensure functioning of the cable up to time t with some probability $1 - \alpha$ can be found from

$$\mathbb{P}\left\{\frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \leq q_{1-\alpha}\right\} \underset{t\to\infty}{\to} 1 - \alpha,$$

where $q_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution. More precisely, we get that the target value is

$$m = q_{1-\alpha} \cdot \frac{\sigma\sqrt{t}}{\mu^{3/2}} + \frac{t}{\mu}.$$

Since the replacements are usually associated with some costs, another frequently encountered problem is choosing a maintenance policy that minimises the expenses. For instance, one might consider replacing the items before failure, since the latter typically induces higher costs. Then, if the cost of replacing a failed item is $c_f > 0$ and that of a working item is $c_w < c_f$, both being constant, the aim is to minimise

$$C_t = c_f \mathbb{E} N_t^{(f)} + c_w \mathbb{E} N_t^{(w)},$$

or, on infinite horizon, $\lim\limits_{t\to\infty} C_t/t$, where $N_t^{(f)}$ and $N_t^{(w)}$ are the number of replacements of failed and nonfailed items before time t, respectively. One of possible strategies can be the so-called block replacement policy, when the items are replaced either at failure or at deterministic times bT, $b \in \mathbb{N}$, $T > 0$. In this case it is known that

$$\lim_{t\to\infty} \frac{C_t}{t} = \frac{c_f \mathbb{E} N_T + c_w}{T}, \tag{1}$$

where N_T is the number of failures at $[0, T]$. Then, if the distribution of inter-arrival times is absolutely continuous, the optimal T minimising (1) satisfies

$$T(\mathbb{E} N_T)' - \mathbb{E} N_T = \frac{c_w}{c_f};$$

for more details, see [1].

It should be mentioned that this policy, as well as other preventive measures, can be effective only if the distribution F of inter-arrival times $\xi_1, \xi_2, \ldots$ has an increasing failure rate. That is, if F is continuous,

$$\frac{F(x+h) - F(x)}{1 - F(x)} \tag{2}$$

should be increasing in x for all $h > 0$ and $x \geq 0$ such that $F(x) < 1$, and in case of a discrete distribution,

$$\frac{\mathbb{P}\{\xi_i = k\}}{\sum\limits_{j=k}^{\infty} \mathbb{P}\{\xi_i = j\}}, \quad i \in \mathbb{N},$$

should be non-decreasing in k, $k \in \mathbb{N} \cup \{0\}$. In other words, the items should wear out over time. Otherwise, the optimal decision is to replace items only upon failure. While this can be the case in many practical instances, it should be noted that the property (2) is violated, in particular, for the renewal process with exponentially distributed inter-arrival times. At the same time, this process turns out to be a very special case of renewal processes, which can be employed in various problems and will be discussed in more detail next week.

References

[1] Barlow, R. and Proschan, F. (1996). *Mathematical theory of reliability.* SIAM.

[2] Janssen, J. and Manca, R. (2006). *Applied semi-Markov processes*. Springer Science & Business Media.