Applications of the Poisson Processes and Related Models

Coursera. Stochastic Processes

June 27, 2021

Renewal theory finds diverse applications in any field involving regenerative systems, such as engineering, population dynamics, genetics and economics. In this overview we would like to present some ideas for applying the special type of renewal processes, the Poisson process, and related models to the real-life problems.

Reliability theory revisited

As it was mentioned before, renewal processes are commonly employed to modelling the failures of some system. A natural problem arising in this context is the choice of a replacement strategy that would minimise the expenses. While in our previous week's text we have briefly addressed this question in the simplest instance when the costs are constant, it should be noted that the quantity, e.g., cost, associated with each renewal can itself be a random variable. In this case, it is common to use the renewal reward process

$$X_t = \sum_{i=1}^{N_t} \xi_i, \quad \xi_1, \xi_2, \dots \sim \text{i.i.d., independent of } N_t, \tag{1}$$

known as the compound Poisson process in a special case when the counting process N_t is Poisson. For instance, the papers [9] and [2] employ this process for modelling the cumulative damage caused to car tyres. In their example, the damage $\xi_1, \xi_2, \dots$ has exponential distribution and is associated with a running distance of trips that occur according to the Poisson process N_t. When the total distance $X_t > 30000$ kilometres, the tyre is considered to be failed. As previously, the authors aim at developing a tyre replacement strategy that would minimise the expected cost; for more details, please refer to the original papers.

Insurance

The process (1) turns out to be particularly important in insurance, where the compound Poisson process constitutes the basis of the celebrated Cramér-Lundberg model, that was introduced in the early XXth century ([8], [3]) and laid the foundation of actuarial risk theory. This model assumes that the claims $\xi_1, \xi_2, \dots$ are i.i.d. positive random variables with non-lattice distribution and finite mean, arriving according to a Poisson process N_t, so that the total claim amount X_t can be described by (1). The generalisation of the Crámer-Lundberg model, allowing for i.i.d. non-exponentially distributed inter-arrival times, was introduced by Andersen [1] and is known as the renewal model.

One of the key issues in this field is maintenance a company's solvency. A common approach to this problem is to consider the risk process R_t defined as

$$R_t = w + rt - X_t,$$

where $w \geq 0$ is a deterministic initial capital and X_t is a cumulative claim amount, and analyse the probability ψ of going bankrupt

$$\psi(w) = \mathbb{P}\{R_t < 0 \text{ for some } t \leq T\}, \quad 0 < T \leq \infty, \quad w \geq 0,$$

known as the ruin probability. Under the assumptions of the Crámer-Lundberg model, such as that of X_t being defined as (1), one obtains that

$$\mathbb{E}R_t = w + (r - \lambda\mu)t,$$

where λ is the intensity of the Poisson process and μ is the mean of the claim size distribution F, leading to the net profit condition $r > \lambda\mu$. Moreover, for this model there is a fundamental Crámer-Lundberg theorem, which states that if $r > \lambda\mu$ and there exists $\nu > 0$, known as the Lundberg exponent, such that

$$\int_0^\infty e^{\nu y}(1 - F(y))\, dy = \frac{r}{\lambda}, \tag{2}$$

the ruin probability ψ is exponentially bounded, i.e.,

$$\psi(w) \leq e^{-\nu w} \qquad \forall w \geq 0.$$

For a special case when F is an exponential distribution, which certainly satisfies (2), there even exists an explicit expression for ψ, namely,

$$\psi(w) = \frac{\lambda\mu}{r} \exp\left\{-\frac{(r - \lambda\mu)w}{r\mu}\right\}, \qquad w \geq 0.$$

Also, an additional condition on F

$$\int_0^\infty y e^{\nu y}(1 - F(y))\, dy = c < \infty$$

allows to obtain also the asymptotic relation

$$\lim_{w \to \infty} e^{\nu w}\psi(w) = \frac{r - \lambda\mu}{\nu c} < \infty.$$

So, the renewal theory allows for a certain characterisation of the ruin probability, which significantly aids the analysis of risks; for more details, see, e.g., [4].

Queueing theory

The last application we would like to cover in this overview is the theory of queues, originating from the renowned paper published in 1909 by A.K. Erlang [5]. While working for the Copenhagen Telephone Exchange, Erlang was looking for a way to optimise its work by determining the number of circuits necessary to maintain the desired service level. To this end, he modelled the calls with the $M/D/1$ system, assuming that they arrive according to a Poisson process N_t and require the same deterministic amount of time T to be served. In 1920, Erlang extended this work to the $M/D/k$ queue [6]. This allowed to obtain the explicit expression for the waiting time distribution, and what is more, pioneered the application of probability theory to the problems of that kind.

Nowadays, the applications of queueing models go far beyond those described by A.K. Erlang. For instance, the queuing theory is widely used in healthcare. As pointed out by [7], there is an empirical evidence in favour of describing the arrival of patients in certain parts of a hospital, such as ICUs, by the Poisson process with some parameter λ, and modelling the service times by the exponential distribution with rate parameter μ. This leads to the $M/M/k$ model, where k represents the number of hospital beds, and allows to easily assess the key performance measures of the healthcare service.

The calculations of exact probabilities in this model are significantly based on the properties of the Poisson process. More precisely, it can be shown that the probability of service delay $\widetilde{p}$ in this case is given by

$$\widetilde{p} = 1 - \sum_{i=0}^{k-1} p_i, \tag{3}$$

where p_i is the probability of having $i = 0, 1, 2, \ldots$ patients defined as

$$p_i = \begin{cases} \dfrac{\lambda^i}{i!\mu^i} p_0, & 1 \leq i \leq k, \\ \dfrac{\lambda^i}{k^{i-k} k! \mu^i} p_o, & i > k, \end{cases} \qquad p_0 = \left(\sum_{j=0}^{k-1} \frac{(rk)^j}{j!} + \frac{r^k k^{k+1}}{k! k(1-r)} \right)^{-1},$$

and $r = \lambda(\mu k)^{-1} < 1$ is the so-called utilisation level, representing the portion of occupied beds. The average waiting time T_w then has the form

$$T_w = \frac{\widetilde{p}}{1 - \mu r k}. \tag{4}$$

The formulae (3) and (4) make the corresponding metrics very easy to compute from data, which allows not only to assess the performance of a medical institution, but also to adjust its capacities in order to achieve the desired service level.

All in all, the renewal processes are a valuable tool for modelling the regenerative systems and analysing their behaviour. Either in a pure form or as a part of some more complicated models, they find a wide range of applications in various areas, from engineering to economics and healthcare.

References

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